Arithmetic and dynamical invariants of the new three-dimensional Keller map

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Abstract

We study the explicit polynomial map $F : \mathbb{A}^3 \to \mathbb{A}^3$ recently announced as a counterexample to the Jacobian conjecture in dimension three. Its fibres are organized by a binary cubic with one fixed coefficient. Over every finite field $\mathbb{F}_q$ of odd characteristic, we determine exactly the number $N_k(q)$ of target points with k rational preimages. We also identify the Jelonek set with the discriminant surface. In characteristic different from 3, its normalization $\mathbb{A}^2 \to S$ is a line-ruled universal homeomorphism, and the omitted locus is exactly $\mathrm{Sing}(S) \simeq \mathbb{G}_m$; in characteristic 3, the surface smooths and the geometric map becomes surjective. In characteristic different from 3,

$$N_3 = \frac{(q-1)(q^2+2)}{6}, \qquad N_0 = \frac{(q-1)(q^2+2)}{3}, \qquad N_1 = \frac{q^3+q^2-2q+2}{2};$$

in characteristic 3 the three quantities are respectively $q^2(q-1)/6$, $q^2(q-1)/3$, and $q^2(q+1)/2$. Thus the limiting fibre statistics are $(N_0, N_1, N_3)/q^3 \to (1/3, 1/2, 1/6)$, the fixed-point statistics of S_3. The discriminant surface has zeta function $Z(T) = (1-q^2T)^{-1}$ in every odd characteristic, and its Grothendieck class is $\mathbb{L}^2$. Over $\mathbb{C}$, the properness complement has the integral homology of a circle and carries the natural degree-three cover with S_3 monodromy.

We also determine the degree growth of every iterate. For the announced ordering of the coordinates, the component-degree vector is

$$\mathbf{d}_n(F) := \begin{pmatrix} \deg(F^n)_1 \\ \deg(F^n)_2 \\ \deg(F^n)_3 \end{pmatrix} = M^{n-1}(7,6,4)^{\mathsf{T}}, \qquad M = \begin{pmatrix} 3 & 3 & 1 \\ 3 & 2 & 1 \\ 3 & 0 & 1 \end{pmatrix},$$

and the first dynamical degree is $3+\sqrt{10}$. For the determinant-one post-composed map $G = (F_3/2, F_2, F_1)$, the corresponding dynamical degree is 5. The difference records that dynamical degree is a conjugacy invariant, not an invariant of independent source and target coordinate changes. The key identities are exact and are accompanied by a reproducible verification script.

1 Introduction

Put $u = 1+xy$ and consider the integer-coefficient polynomial map $F = (A, B, C) : \mathbb{A}^3 \to \mathbb{A}^3$ given by

$$\begin{aligned} A &= u^3 z + y^2 u(4+3xy), \\ B &= y + 3xu^2 z + 3xy^2(4+3xy), \\ C &= 2x - 3x^2 y - x^3 z. \end{aligned} \tag{1}$$

A direct determinant calculation gives

$$\det JF = -2,$$

and the three distinct points

$$(0, 0, -1/4), \qquad (1, -3/2, 13/2), \qquad (-1, 3/2, 13/2)$$

all map to $(-1/4, 0, 0)$. Thus over every field of characteristic zero the map is Keller and noninjective. The example was publicly announced by Alpöge on 19–20 July 2026, with discovery credit to Fable [1]; independent derivations and structural analyses appeared immediately afterward [10, 18, 24, 19, 20].

The purpose of this note is deliberately narrow. We use the already-public binary-cubic description of the fibres as a gateway to three calculations and syntheses:

1. a normalization-level description of the Jelonek surface, including its line ruling, universal-homeomorphism structure, motivic class, and the geometric transition in characteristic 3;

2. the complete rational fibre enumerator over every finite field of odd characteristic, including the exceptional formulas in characteristic 3; and

3. the exact degree sequence of every iterate, for both the announced coordinate ordering and a natural determinant-one post-composition.

The arithmetic result is an exact finite-field realization of the factorization statistics suggested by the generic S_3 monodromy. It sits within the literature on factorization in linear families and with prescribed coefficients [4, 17, 11, 21, 6], while the permutation-group viewpoint is part of the theory of exceptional covers [9]. Jelonek's hypersurface and uniruledness theorems provide the general setting for the nonproperness calculation [13, 23, 14]. The dynamical result exhibits two elementary invariant max-plus chambers, so no asymptotic approximation or generic-cancellation assumption is required. The standard existence and birational invariance theory for dynamical degrees may be found in [5, 3]; the compactification viewpoint is developed in dimension two in [7], and the matrix-controlled degree growth of monomial maps is studied in [12, 8].

The map has good reduction as an *étale* polynomial map in every odd characteristic, because its Jacobian determinant remains -2. In characteristic 2 the Keller condition disappears. This is the only characteristic excluded from the arithmetic theorem.

2 The binary-cubic gateway

For target coordinates (a, b, c) define the binary cubic

$$H_{a,b,c}(R, T) = 2aR^3 - bR^2T + 2RT^2 - cT^3. \tag{2}$$

Its discriminant is

$$\mathrm{Disc}(H_{a,b,c}) = -4J(a, b, c), \qquad J = 27a^2c^2 - 18abc + b^3c + 16a - b^2. \tag{3}$$

This cubic and the associated S_3 monodromy and discriminant surface were already identified in the same-day public analyses cited above. We recall the precise root–fibre correspondence needed for counting. For the foundational theory of nonproperness sets of polynomial maps, see [13].

Proposition 2.1 (Simple-root correspondence)**.** *Let k be a field of characteristic different from 2. For every $(a,b,c) \in k^3$, the k-rational points of the fibre $F^{-1}(a,b,c)$ are in canonical bijection with the simple k-rational projective roots of $H_{a,b,c}$ in $\mathbb{P}^1(k)$.*

Proof. For an input (x,y,z) set $u = 1+xy$. Substitution in (1) gives the polynomial identities

$$2Ax^3 - Bx^2u + 2xu^2 - Cu^3 = 0, \qquad 3Ax^2 - Bxu + u^2 = u. \tag{4}$$

The first identity says that $[R:T] = [x:u]$ is a projective root of $H_{A,B,C}$. If $u \neq 0$, write $r = x/u$. Then the second identity is

$$\frac{1}{2}\frac{d}{dr}H_{A,B,C}(r,1) = 3Ar^2 - Br + 1 = \frac{1}{u} \neq 0,$$

so the root is simple. If $u = 0$, then $xy = -1$ and $B = -2y = 2/x \neq 0$. The derivative at the root $[1:0]$ is $-B$, so that root is simple as well.

Conversely, let $[R:T]$ be a simple rational root. First suppose it is the finite root $r = R/T$ and put

$$s = 3ar^2 - br + 1.$$

Simplicity means $s \neq 0$. If $r \neq 0$, the unique inverse point is

$$x = \frac{r}{s}, \qquad y = \frac{1-s}{r}, \qquad z = as^3 - s(s+3)y^2. \tag{5}$$

Indeed, the definitions of s and of the root imply

$$b = \frac{1-s}{r} + 3ra, \qquad c = r(1+s) - r^3a,$$

and direct substitution in (1) gives $(A,B,C) = (a,b,c)$. If $r = 0$, the root equation gives $c = 0$, and its unique inverse point is

$$(x,y,z) = (0,b,a-4b^2).$$

Finally, a root at infinity forces $a = 0$, and simplicity forces $b \neq 0$. Its unique inverse point is obtained by setting

$$x = \frac{2}{b}, \qquad y = -\frac{b}{2}, \qquad z = \frac{5x-c}{x^3}.$$

These three inverse charts prove both surjectivity and uniqueness of the correspondence. □

Remark 2.2. Multiple roots of the binary cubic do not represent ramification points of F; they represent would-be inverse points that have escaped to infinity. This is why a polynomial map with constant nonzero Jacobian can have a nonconstant fibre cardinality.

2.1 Geometric image and the Jelonek surface

Let k be algebraically closed of characteristic different from 2, and let S_F denote the nonproperness (Jelonek) set of F_k. The general theory says that the nonempty nonproperness set of a generically finite polynomial map is a hypersurface and is uniruled [13, 23, 14, 15]. Here both the hypersurface and its ruling are explicit.

Proposition 2.3 (Jelonek set, image, and smoothing)**.** *One has*

$$S_F = S := V(J) \subset \mathbb{A}^3_k.$$

If $\operatorname{char} k \neq 3$*, put*

$$\Gamma = V(27ac^2 - 4,\ 3bc - 4).$$

Then $\Gamma \simeq \mathbb{G}_m$ *and*

$$\#F^{-1}(a,b,c) = \begin{cases} 3, & (a,b,c) \notin S, \\ 1, & (a,b,c) \in S - \Gamma, \\ 0, & (a,b,c) \in \Gamma. \end{cases}$$

Consequently

$$F(\mathbb{A}^3_k) = \mathbb{A}^3_k - \Gamma, \qquad \Gamma = \operatorname{Sing}(S).$$

If $\operatorname{char} k = 3$*, then* S *is smooth,* $F : \mathbb{A}^3_k \to \mathbb{A}^3_k$ *is surjective, and its geometric fibres have size* 3 *off* S *and size* 1 *on* S*.*

Proof. Over an algebraically closed field, a squarefree binary cubic has three simple projective roots, a double-plus-simple cubic has one, and a triple-root cubic has none. Proposition 2.1 therefore gives all the displayed fibre sizes. Off S, the finite scheme of projective roots is finite étale of degree three, and the inverse charts in Proposition 2.1 identify it with the inverse image of the target. Thus F is proper over the complement of S. At a point of S, a multiple root can be split along a transverse one-parameter deformation. In the finite-root chart the reconstructing denominator is the derivative of the cubic; it tends to zero, while the root tends to a nonzero value (zero cannot be multiple because the derivative there is 2), so x tends to infinity. For a multiple root at $[1:0]$, one has $b = 0$ and the infinity chart $x = 2/b$ gives the same conclusion. The valuative criterion therefore places every point of S in S_F, proving $S_F = S$.

In characteristic different from 3, comparison with a cube of a linear form shows that the triple-root locus is exactly Γ; the coordinate c is nonzero there and identifies Γ with $\mathbb{G}_m$. Direct differentiation, or the cusp coordinates in Proposition 2.4 below, gives $\operatorname{Sing}(S) = \Gamma$. In characteristic 3 a cube of a linear form has zero RT^2 coefficient, whereas (2) has coefficient $2 \neq 0$. Hence a triple root is impossible. Moreover $J = a + b^3c - b^2$ and $\partial J/\partial a = 1$, proving both smoothness and surjectivity. □

The equality between the omitted curve and the singular locus was already recorded in the same-day construction analysis [20]. The next proposition packages it with the normalization and makes the characteristic-3 change transparent.

Proposition 2.4 (Normalization, ruling, and motivic class)**.** *Over every field* k *of odd characteristic, the polynomial map*

$$\nu : \mathbb{A}^2_{w,c} \longrightarrow S, \qquad (w,c) \longmapsto (w^2 - cw^3,\ 4w - 3cw^2,\ c) \tag{6}$$

is a finite universal homeomorphism. If $\operatorname{char} k \neq 3$*, it is the normalization and is not an isomorphism; if* $\operatorname{char} k = 3$*, it is an isomorphism. Its vertical lines exhibit* S *as covered by affine lines. Furthermore,*

$$[S] = \mathbb{L}^2 \quad \text{in } K_0(\mathrm{Var}_k).$$

Over $\mathbb{C}$*, the analytic space* $S(\mathbb{C})$ *is homeomorphic to* $\mathbb{C}^2$*, although it is singular along* Γ*.*

Proof. Direct substitution gives $J \circ \nu = 0$, and on the image

$$w^2 - bw + 3a = 0. \tag{7}$$

Thus w is integral over the coordinate ring of S, so ν is finite. Introduce

$$X = 4 - 3bc = (2 - 3cw)^2, \qquad Y = 27ac^2 + 8 - 9bc = (2 - 3cw)^3.$$

The exact polynomial identity

$$Y^2 - X^3 = 27c^2 J \tag{8}$$

shows that, on $c \neq 0$ in characteristic different from 3, the surface is the cusp $Y^2 = X^3$ times $\mathbb{G}_m$. The parametrization $t \mapsto (t^2, t^3)$ is bijective after every field extension: for $X \neq 0$ its inverse is $t = Y/X$, and the origin has the unique preimage $t = 0$. When $c = 0$ one recovers $w = b/4$. Hence ν is universally injective and surjective; being finite, it is a universal homeomorphism. It is birational, and the normality of $\mathbb{A}^2$ makes it the normalization. In characteristic 3, S is the graph $a = b^2 - b^3 c$ and $\nu(w, c) = (w^2 - cw^3, w, c)$ is visibly an isomorphism.

For fixed w, formula (6) is affine-linear in c, which gives the asserted ruling by lines. For the motivic identity, the slice $c = 0$ is an affine line. On $c \neq 0$, the cusp decomposes into its origin and a copy of $\mathbb{G}_m$, so it has class $\mathbb{L}$; therefore

$$[S] = \mathbb{L} + \mathbb{L}(\mathbb{L} - 1) = \mathbb{L}^2.$$

The same formula is immediate from $S \simeq \mathbb{A}^2$ in characteristic 3. Finally, over $\mathbb{C}$ a finite bijective morphism is a proper continuous bijection on analytic points and hence a homeomorphism. The corresponding topological invariance of the finite étale site is a general property of universal homeomorphisms [22]. □

In the terminology of [14, 15], the displayed line ruling says more precisely that S_F has degree of uniruledness 1, the smallest possible value; the general bounds in those papers depend on the degree of the polynomial map.

Corollary 2.5 (Topology of the properness complement)**.** *Let $U = \mathbb{C}^3 - S(\mathbb{C})$. Then U has the integral homology of a circle:*

$$H_i(U, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, 1, \\ 0, & i \geq 2. \end{cases}$$

The restriction $F^{-1}(U) \to U$ is a finite étale cover of degree three. Its geometric monodromy is S_3.

Proof. Proposition 2.4 gives a homeomorphism $S(\mathbb{C}) \simeq \mathbb{C}^2$. Its one-point compactification is therefore S^4. Alexander duality in the one-point compactification S^6 of $\mathbb{C}^3$ gives the displayed homology. The covering statement follows from the squarefree root scheme and Proposition 2.1; the S_3 monodromy of the generic cubic was computed in the immediate structural analysis [19]. □

3 Exact finite-field fibre distributions

Let q be an odd prime power. For $k \in \{0, 1, 3\}$ write

$$N_k(q) = \#\{(a, b, c) \in \mathbb{F}_q^3 : \#F^{-1}(a, b, c)(\mathbb{F}_q) = k\}.$$

Theorem 3.1 (Rational fibre enumerator)**.** *There are no rational fibres of size other than* 0*,* 1*, or* 3*. If* $\operatorname{char} \mathbb{F}_q \neq 3$*, then*

$$\boxed{\begin{aligned} N_3(q) &= \frac{(q-1)(q^2+2)}{6}, \\ N_0(q) &= \frac{(q-1)(q^2+2)}{3}, \\ N_1(q) &= \frac{q^3+q^2-2q+2}{2}. \end{aligned}} \tag{9}$$

If $\operatorname{char} \mathbb{F}_q = 3$*, then*

$$\boxed{N_3(q) = \frac{q^2(q-1)}{6}, \qquad N_0(q) = \frac{q^2(q-1)}{3}, \qquad N_1(q) = \frac{q^2(q+1)}{2}.} \tag{10}$$

Proof. By Proposition 2.1, fibre size is the number of simple rational projective roots of $H_{a,b,c}$. A cubic has 0, 1, or 3 such roots, but never exactly 2.

A target has three preimages precisely when its cubic has three distinct rational projective roots. Each three-element subset of $\mathbb{P}^1(\mathbb{F}_q)$ determines a binary cubic up to nonzero scalar. It can be scaled uniquely to the form (2) exactly when its RT^2 coefficient is nonzero. We therefore count the three-element subsets for which that coefficient vanishes.

If the subset contains infinity and two finite roots r, s, its cubic is proportional to

$$T(R - rT)(R - sT),$$

whose RT^2 coefficient vanishes exactly when $r + s = 0$. There are $(q-1)/2$ such unordered pairs.

For three distinct finite roots r, s, t, the bad condition is

$$rs + rt + st = 0. \tag{11}$$

Choose the ordered pair (r, s). Distinctness and the impossibility of $r + s = 0$ leave $(q-1)^2$ pairs, and then

$$t = -\frac{rs}{r+s}$$

is forced. If the characteristic is not 3, the condition $t = r$ occurs when either $r = 0$ or $r = -2s$, and $t = s$ occurs symmetrically. These four disjoint coincidence families each have $q-1$ elements. Hence the number of ordered distinct solutions of (11) is $(q-1)(q-5)$, and the number of unordered solutions is $(q-1)(q-5)/6$. Including the subsets containing infinity, the total number of bad subsets is

$$\frac{(q-1)(q-5)}{6} + \frac{q-1}{2} = \frac{(q-1)(q-2)}{6}.$$

Subtracting from $\binom{q+1}{3}$ gives

$$N_3(q) = \binom{q+1}{3} - \frac{(q-1)(q-2)}{6} = \frac{(q-1)(q^2+2)}{6}.$$

In characteristic 3, the alternatives $r = -2s$ and $s = -2r$ are precisely the already-excluded equality $r = s$. Only the families $r = 0$ and $s = 0$ remain, so the affine bad count becomes $(q-1)(q-3)/6$. The total bad count is then

$$\frac{(q-1)(q-3)}{6} + \frac{q-1}{2} = \frac{q(q-1)}{6},$$

and

$$N_3(q) = \binom{q+1}{3} - \frac{q(q-1)}{6} = \frac{q^2(q-1)}{6}.$$

It remains to recover N_0 and N_1. Counting target points and domain points gives

$$N_0 + N_1 + N_3 = q^3, \qquad N_1 + 3N_3 = q^3.$$

Consequently $N_0 = 2N_3$, and substitution gives all the displayed formulas. □

Corollary 3.2 (Value-set size and limiting statistics)**.** *If* $\operatorname{char} \mathbb{F}_q \neq 2, 3$*, then*

$$\#F(\mathbb{F}_q^3) = N_1(q) + N_3(q) = \frac{2q^3 + q^2 - 2q + 2}{3}.$$

In characteristic 3*,*

$$\#F(\mathbb{F}_q^3) = \frac{q^2(2q+1)}{3}.$$

In either case,

$$\frac{1}{q^3}(N_0, N_1, N_3) \longrightarrow \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right).$$

These are the proportions of elements of S_3 *having respectively* 0*,* 1*, and* 3 *fixed points in its natural permutation action.*

The count can be refined by factorization type. This also isolates the small correction supported on the discriminant surface.

Corollary 3.3 (Factorization stratification)**.** *For odd* q *with* $\operatorname{char} \mathbb{F}_q \neq 3$*, the target cubics have the following distribution.*

Factorization type in $\mathbb{P}^1(\mathbb{F}_q)$	*Number of targets*
three distinct rational roots	$(q-1)(q^2+2)/6$
one rational root and an irreducible quadratic	$q^2(q-1)/2$
one double root and one simple root	$q^2 - q + 1$
irreducible cubic	$(q-1)(q^2-1)/3$
one triple root	$q - 1$

In characteristic 3*, the five corresponding counts are*

$$\frac{q^2(q-1)}{6}, \qquad \frac{q^2(q-1)}{2}, \qquad q^2, \qquad \frac{q^2(q-1)}{3}, \qquad 0.$$

Proof. Proposition 2.4 makes ν bijective on points over every finite extension, so $J = 0$ has q^2 rational points.

If the characteristic is not 3, a triple-root cubic with fixed nonzero RT^2 coefficient is determined by a nonzero finite root, giving $q-1$ targets. In characteristic 3, cubes of linear forms have zero RT^2 coefficient, so no triple root occurs. Subtracting these counts from the discriminant count gives the double-plus-simple row. The remaining rows follow from Theorem 3.1: the squarefree $1+2$ type and the double-plus-simple type contribute to N_1, while irreducible and triple-root cubics contribute to N_0. □

Remark 3.4 (Geometric omission versus arithmetic omission). Suppose first that $\operatorname{char}\mathbb{F}_q \neq 3$. The $q-1$ triple-root targets are exactly $\Gamma(\mathbb{F}_q)$, so they remain omitted after passage to the algebraic closure. The other

$$\frac{(q-1)(q^2-1)}{3}$$

targets counted by N_0 have irreducible cubics: they acquire three geometric preimages, but none is rational over $\mathbb{F}_q$. In characteristic 3 the geometric omitted locus is empty, and every target counted by N_0 is of this second, purely arithmetic kind. Thus rational non-surjectivity persists even when geometric surjectivity appears.

Corollary 3.5 (Zeta functions). *Let $S = V(J) \subset \mathbb{A}^3_{\mathbb{F}_q}$ for an odd prime power q. With*

$$Z(X/\mathbb{F}_q, T) = \exp\left(\sum_{m\geq 1} \#X(\mathbb{F}_{q^m})\frac{T^m}{m}\right),$$

one has

$$\boxed{Z(S/\mathbb{F}_q, T) = \frac{1}{1-q^2T}, \qquad Z((\mathbb{A}^3 - S)/\mathbb{F}_q, T) = \frac{1-q^2T}{1-q^3T}.}$$

Proof. The universal homeomorphism in Proposition 2.4 is bijective over every finite extension. Consequently,

$$\#S(\mathbb{F}_{q^m}) = q^{2m}.$$

The first identity follows from

$$\exp\left(\sum_{m\geq 1} q^{2m}\frac{T^m}{m}\right) = \frac{1}{1-q^2T}.$$

Subtracting from $\#\mathbb{A}^3(\mathbb{F}_{q^m}) = q^{3m}$ gives the second. □

Remark 3.6 (The characteristic-3 smoothing). Modulo 3,

$$J = a + b^3c - b^2,$$

so S is the smooth graph $a = b^2 - b^3c \simeq \mathbb{A}^2$; indeed $\partial J/\partial a = 1$. The triple-root stratum, the omitted curve, and the singular locus therefore disappear simultaneously, even though $\det JF = -2 = 1$ and the map remains *étale*. The same zeta function in Corollary 3.5 thus arises from an actual affine plane in characteristic 3 and from a nonnormal cuspidal surface universally homeomorphic to an affine plane in the other odd characteristics.

4 Exact degree growth

We now work over $\mathbb{C}$ and regard F as a polynomial self-map by identifying its ordered target coordinates (A, B, C) with the ordered source coordinates (x, y, z). For a polynomial self-map H, its algebraic degree is $\deg H = \max_i \deg H_i$. For $n \geq 1$, write

$$(\alpha_n, \beta_n, \gamma_n) = \big(\deg(F^n)_1, \deg(F^n)_2, \deg(F^n)_3\big).$$

Theorem 4.1 (Degree recurrence for the announced ordering)**.** *For every $n \geq 1$,*

$$\begin{pmatrix} \alpha_n \\ \beta_n \\ \gamma_n \end{pmatrix} = \begin{pmatrix} 3 & 3 & 1 \\ 3 & 2 & 1 \\ 3 & 0 & 1 \end{pmatrix}^{n-1} \begin{pmatrix} 7 \\ 6 \\ 4 \end{pmatrix}. \tag{12}$$

Each component sequence d_n satisfies

$$d_{n+2} = 6d_{n+1} + d_n \qquad (n \geq 1). \tag{13}$$

In particular,

$$\delta_1(F) = \lim_{n\to\infty} \deg(F^n)^{1/n} = 3 + \sqrt{10}.$$

Proof. Suppose an iterate has component degrees (α, β, γ) and each component has a unique top homogeneous monomial. Tropicalizing (1) gives

$$\begin{aligned} \alpha' &= \max(3\alpha + 3\beta + \gamma,\ 2\alpha + 4\beta), \\ \beta' &= \max(3\alpha + 2\beta + \gamma,\ 2\alpha + 3\beta), \\ \gamma' &= \max(3\alpha + \gamma,\ 2\alpha + \beta). \end{aligned}$$

The difference between the first and second candidate in all three lines is

$$h = \alpha - \beta + \gamma.$$

For F itself, the component degrees are $(7, 6, 4)$, so $h = 5 > 0$, and the unique top monomials are

$$x^3y^3z, \qquad 3x^3y^2z, \qquad -x^3z.$$

Thus the first branch is strictly dominant. On this branch the degree vector is multiplied by

$$M = \begin{pmatrix} 3 & 3 & 1 \\ 3 & 2 & 1 \\ 3 & 0 & 1 \end{pmatrix},$$

and the new chamber functional is

$$h' = \alpha' - \beta' + \gamma' = 3\alpha + \beta + \gamma > 0.$$

The chamber is therefore invariant. Moreover, composing the three unique top monomials produces a unique top monomial at the next stage, so there is no possible leading-term cancellation. Induction proves (12).

The characteristic polynomial of M is $t(t^2 - 6t - 1)$. Directly,

$$M^2(7, 6, 4)^{\mathsf{T}} = 6M(7, 6, 4)^{\mathsf{T}} + (7, 6, 4)^{\mathsf{T}},$$

and multiplying this identity by powers of M proves (13). Finally α_n is always the largest component degree, since on the invariant branch $\alpha' - \beta' = \beta$ and $\beta' - \gamma' = 2\beta$. The dominant root of $t^2 - 6t - 1$ is $3 + \sqrt{10}$. □

The first five component-degree vectors are

$$\begin{aligned}&(7,6,4),\quad (43,37,25),\quad (265,228,154),\\&(1633,1405,949),\quad (10063,8658,5848).\end{aligned}\tag{14}$$

There is a second natural self-map obtained by post-composing the same Keller map with a linear target isomorphism. Let

$$L(a,b,c)=(c/2,b,a),\qquad G=L\circ F=(C/2,B,A).$$

Since $\det L=-1/2$, the normalized map satisfies $\det JG=1$.

Theorem 4.2 (Degree recurrence after determinant-one post-composition)**.** *Write*

$$(p_n,q_n,r_n)=\big(\deg(G^n)_1,\deg(G^n)_2,\deg(G^n)_3\big).$$

Then

$$\begin{pmatrix}p_n\\q_n\\r_n\end{pmatrix}=\begin{pmatrix}3&0&1\\3&2&1\\3&3&1\end{pmatrix}^{n-1}\begin{pmatrix}4\\6\\7\end{pmatrix},\tag{15}$$

and explicitly

$$p_n=\frac{15\cdot 5^{n-1}+1}{4},\qquad q_n=\frac{25\cdot 5^{n-1}-1}{4},\qquad r_n=\frac{15\cdot 5^{n-1}-1}{2}.\tag{16}$$

Consequently $\delta_1(G)=5$.

Proof. The unique leading monomials of G occur in the order

$$-\tfrac{1}{2}x^3z,\qquad 3x^3y^2z,\qquad x^3y^3z,$$

with component degrees $(4,6,7)$. The same chamber functional $p-q+r$ is positive, and the dominant degree branches give the matrix

$$N=\begin{pmatrix}3&0&1\\3&2&1\\3&3&1\end{pmatrix}.$$

As before, the transformed chamber functional is $3p+q+r>0$, so induction proves (15) without cancellation. The characteristic polynomial of N is $t(t-1)(t-5)$, and solving from the initial vector $(4,6,7)$ gives (16). The third component is largest because $q'-p'=2q$ and $r'-q'=q$, hence the first dynamical degree is 5. □

Corollary 4.3 (Topological degree)**.** *Both F and G have generic fibre degree* 3*. Consequently their top dynamical degree is*

$$\delta_3(F)=\delta_3(G)=3,$$

and the generic fibre degree of the nth iterate is 3^n.

Proof. On the dense open set $J\neq 0$, Proposition 2.1 identifies the geometric fibre with the three distinct roots of the binary cubic. Linear post-composition does not change generic degree, and generic degrees multiply under composition. For a dominant rational self-map of a threefold, this generic degree is the top dynamical degree. □

Remark 4.4 (Why the two answers differ). The values $3+\sqrt{10}$ and 5 are not competing invariants of a single rational self-map. Dynamical degree is invariant under conjugacy $H \circ F \circ H^{-1}$. Here $G = L \circ F$ is obtained only by post-composition. Iteration requires an identification of target coordinates with source coordinates, and changing that identification can change degree growth. Thus neither value should be presented as an invariant under ordinary left–right equivalence of polynomial maps.

5 Verification and scope

The accompanying script `verify.py` performs the following exact checks:

- $\det JF = -2$ and the displayed three-point collision;
- the two identities (4) and all three inverse charts in Proposition 2.1;
- the discriminant, cusp identity (8), normalization relation (7), singular locus, triple-root parametrization, and characteristic-3 smoothing;
- the component degrees of F, F^2, G, and G^2, both matrix characteristic polynomials, the scalar recurrences, and several subsequent degree vectors; and
- exhaustive fibre histograms over the prime fields $\mathbb{F}_3, \mathbb{F}_5, \mathbb{F}_7, \mathbb{F}_{11}, \mathbb{F}_{13}$, and $\mathbb{F}_{17}$; and
- independent quadratic-extension checks over $\mathbb{F}_9$ and $\mathbb{F}_{25}$.

The finite-field enumeration is an independent check, not the proof of Theorem 3.1; the proof applies uniformly to all odd prime powers.

This paper does not claim novelty for the binary cubic, its discriminant, the generic S_3 monodromy, or the complex image and nonproperness set; all were placed in the public record in the immediate analyses [10, 18, 24, 19, 20]. The novelty claim is limited to the exact finite-field distributions and exact degree-growth calculations for this map, together with the synthesis connecting the normalization, line ruling, universal-homeomorphism structure, zeta function, and characteristic-3 transition, subject to the provisional nature of a same-day literature review. General exact and asymptotic frameworks for factorization patterns in coefficient-restricted families already exist [4, 17, 11, 21, 6]; the elementary cubic count here is not claimed as a new general counting method. Likewise, the universal-homeomorphism formalism and its invariance properties are standard [22]; the contribution is their explicit realization in this Keller map.

Natural next problems include the value distributions for the higher-degree families appearing in [10, 20], the arithmetic of iterated value sets, the second dynamical degree, a compactification on which the degree matrices above arise geometrically, and whether the coincidence between omitted, singular, and nonnormal loci persists in any broader class of nonproper étale polynomial maps.

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